Factor Models for Multiple Time Series

Qiwei Yao

Department of Statistics, London School of Economics

q.yao@lse.ac.uk

Joint work with

Neil Bathia, University of Melbourne

Clifford Lam, London School of Mathematics

Jiazhu Pan, University of Strathclyde

Flavio Ziegelmann, Federal University of Rio Grande do Sul

- Econometric factor models: a brief survey
- Statistical factor models: identification
- Estimation
    - expanding white noise space: non-stationary factors
    - eigenanalysis: stationary cases
- Asymptotic properties (stationary cases only in this talk)
    - fixed p: fast convergence rate for zero-eigenvalues
    - $p \to \infty$: convergence rates independent of p
- Illustration with real data sets
    - temperature data
    - implied volatility surfaces
    - densities of intraday returns

Econometric modelling: represent a $p \times 1$ time series $\mathbf{y}_t$ as

$$\mathbf{y}_t = \mathbf{f}_t + \boldsymbol{\xi}_t,$$

both $\mathbf{f}_t$ and $\boldsymbol{\xi}_t$ are <u>unobservable</u>, and

- $\mathbf{f}_t$: driven by r common factors, and $r << p$
- $\boldsymbol{\xi}_t$: idiosyncratic components

Basic idea. The dynamical structure of each component of $\mathbf{y}_t$ is driven by the r common factors plus one or a few idiosyncratic components.

Practical motivation: asset pricing models, yield curves, portfolio risk management, derivative pricing, macroeconomic behaviour and forecasting, consumer theory etc.

Sargent & Sims (1977) and Geweke (1977): dynamic-factor models

Chamberlain & Rothschild (1983): *approximate* and *static* factor models

Forni, Hallin, Lippi & Reichlin (2002 –): generalized dynamic factor models – combining the above two together

$$y_{it} = b_{i1}(L)u_{1t} + \cdots + b_{ir}(L)u_{rt} + \xi_{it}, \quad i = 1, 2, \cdots, \ t = 0, \pm 1, \cdots,$$

- $u_{kt} \sim \mathrm{WN}(0, 1)$, $k = 1, \cdots, r$, are common (dynamic) factors, and are uncorrelated with each other,
- ξ_{it} are stationary in t, are idiosyncratic noise, and $\{u_{kt}\}$ and $\{\xi_{it}\}$ are uncorrelated.

Only y_{it} are observable.

Let $\boldsymbol{\xi}_{pt} = (\xi_{1t}, \cdots, \xi_{pt})^\tau$ and $\mathbf{y}_{pt} = (y_{1t}, \cdots, y_{pt})^\tau$.

Assumption: As $p \to \infty$, it holds almost surely on $[-\pi,\ \pi]$ that all the eigenvalues of spectral density matrices of $\boldsymbol{\xi}_{pt}$ are uniformly bounded, and only the r largest eigenvalues of $(\mathbf{y}_{pt} - \boldsymbol{\xi}_{pt})$ converge to ∞.

Intuition: The r common factors affect the dynamics of most component series, while each idiosyncratic noise only affects the dynamics of a few component series.

Characteristics result: As $p \to \infty$, it holds almost surely on $[-\pi,\ \pi]$ that all the r largest eigenvalues of spectral density matrices of $\mathbf{y}_{pt}$ converge to ∞, and the $(r+1)$-th largest eigenvalue is uniformly bounded.

The model is asymptotically identifiable, when the number of

- Estimation for GDFM when r is given — Dynamic principle component analysis (Brillinger 1981):

  i. Obtain an estimator $\widehat{\mathbf{\Sigma}}(\theta)$ for spectral density matrix of $\mathbf{y}_t$, $\theta \in [-\pi, \pi]$

  ii. Find eigenvalues and eigenvectors of $\widehat{\mathbf{\Sigma}}(\theta)$

  iii. Project $\mathbf{y}_t$ onto the space spanned by the r eigenvectors corresponding to the r largest eigenvalues:

  the projection is defined as the mean square limit of a Fourier sequence, and

  each component of the projection is a sum of r uncorrelated MA processes.

- Determine r: only identifiable when $p \to \infty$!

  'There is no way a slowly diverging sequence can be told from an eventually bounded sequence' (Forni et al. 2000).

Let $\{\mathbf{y}_t\}$ be a $p \times 1$ time series defined by

$$\mathbf{y}_t = \mathbf{A}\mathbf{x}_t + \boldsymbol{\varepsilon}_t,$$

Let $\{\mathbf{y}_t\}$ be a $p \times 1$ time series defined by

$$\mathbf{y}_t = \mathbf{A}\mathbf{x}_t + \boldsymbol{\varepsilon}_t,$$

$\mathbf{x}_t$: $r \times 1$ unobservable factors, $r\ (< p)$ unknown

$\mathbf{A}$: $p \times r$ unknown constant factor loading matrix

$\{\boldsymbol{\varepsilon}_t\}$: vector WN($\boldsymbol{\mu}_\varepsilon, \boldsymbol{\Sigma}_\varepsilon$)

no linear combinations of $\mathbf{x}_t$ are WN.

Let $\{\mathbf{y}_t\}$ be a $p \times 1$ time series defined by

$$\mathbf{y}_t = \mathbf{A}\mathbf{x}_t + \boldsymbol{\varepsilon}_t,$$

$\mathbf{x}_t$: $r \times 1$ unobservable factors, $r\ (< p)$ unknown

$\mathbf{A}$: $p \times r$ unknown constant factor loading matrix

$\{\boldsymbol{\varepsilon}_t\}$: vector WN($\boldsymbol{\mu}_\varepsilon, \boldsymbol{\Sigma}_\varepsilon$)

no linear combinations of $\mathbf{x}_t$ are WN.

Lack of identification: $(\mathbf{A}, \mathbf{x}_t)$ may be replaced by $(\mathbf{A}\mathbf{H}, \mathbf{H}^{-1}\mathbf{x}_t)$ for any invertible $\mathbf{H}$.

Therefore, we assume $\mathbf{A}^\tau \mathbf{A} = \mathbf{I}_r$

But factor loading space $\mathcal{M}(\mathbf{A})$ is uniquely defined

Let $\{\mathbf{y}_t\}$ be a $p \times 1$ time series defined by

$$\mathbf{y}_t = \mathbf{A}\mathbf{x}_t + \boldsymbol{\varepsilon}_t,$$

$\mathbf{x}_t$: $r \times 1$ unobservable factors, r $(< p)$ unknown

$\mathbf{A}$: $p \times r$ unknown constant

What is new?

- No distributional assumption on ε_t. More significantly, allow correlation between ε_t and $\mathbf{x}_{t+k}$.

What is new?

- No distributional assumption on ε_t. More significantly, allow correlation between ε_t and $\mathbf{x}_{t+k}$.
- Factor $\mathbf{x}_t$, and therefore also $\mathbf{y}_t$, may be nonstationary, not necessarily driven by unit roots.

What is new?

- No distributional assumption on $\boldsymbol{\varepsilon}_t$. More significantly, allow correlation between $\boldsymbol{\varepsilon}_t$ and $\mathbf{x}_{t+k}$.
- Factor $\mathbf{x}_t$, and therefore also $\mathbf{y}_t$, may be nonstationary, not necessarily driven by unit roots.
- A new estimation method for stationary cases: an eigenanalysis, applicable when $p >> n$

What is new?

- No distributional assumption on ε_t. More significantly, allow correlation between ε_t and $\mathbf{x}_{t+k}$.
- Factor $\mathbf{x}_t$, and therefore also $\mathbf{y}_t$, may be nonstationary, not necessarily driven by unit roots.
- A new estimation method for stationary cases: an eigenanalysis, applicable when $p >> n$

Key: estimate $\mathbf{A}$, or more precisely, $\mathcal{M}(\mathbf{A})$.

With available an estimator $\widehat{\mathbf{A}}$, a natural estimator for factor and the associated residuals are

$$\widehat{\mathbf{x}}_t = \widehat{\mathbf{A}}^\tau \mathbf{y}_t, \qquad \widehat{\varepsilon}_t = (\mathbf{I}_p - \widehat{\mathbf{A}}\widehat{\mathbf{A}}^\tau)\mathbf{y}_t.$$

By modelling the lower-dimensional $\widehat{\mathbf{x}}_t$, we obtain the dynamical model for $\mathbf{y}_t$:

$$\widehat{\mathbf{y}}_t = \widehat{\mathbf{A}}\widehat{\mathbf{x}}_t.$$

Reconciling to econometric models

'Common factors' & 'idiosyncratic noise': conceptually appealing, only identifiable when $p \to \infty$.

Goal: identify those components of $\mathbf{x}_t$, each of them affects most (or a few) components of $\mathbf{y}_t$.

Put $\mathbf{A} = (\mathbf{a}_1, \cdots, \mathbf{a}_r)$ and $\mathbf{x}_t = (x_{t1}, \cdots, x_{tr})'$. Then

$$\mathbf{y}_t = \mathbf{a}_1 x_{t1} + \cdots + \mathbf{a}_r x_{tr} + \boldsymbol{\varepsilon}_t.$$

Hence the number of non-zero coefficients of $\mathbf{a}_j$ is the number of components of $\mathbf{y}_t$ which are affected by the factor x_{tj}.

To avoid the correlation among the components of $\mathbf{x}_t$, apply PCA to $\mathbf{x}_t$, i.e. replace $(\mathbf{A}, \mathbf{x}_t)$ by $(\mathbf{A\Gamma}, \mathbf{\Gamma}'\mathbf{x}_t)$, where $\mathbf{\Gamma}$ is an $r \times r$ orthogonal matrix defined in Var$(\mathbf{x}_t) = \mathbf{\Gamma D \Gamma}'$.

Eigenvalues of Var$(\mathbf{x}_t)$ are different, this representation is unique.

Lemma 1. Let $\mathbf{A}_1\mathbf{z}_1 = \mathbf{A}_2\mathbf{z}_2$, where, for $i = 1, 2$, $\mathbf{A}_i$ is $p \times r$ matrix, $\mathbf{A}_i'\mathbf{A}_i = \mathbf{I}_r$, and $\mathbf{z}_i = (z_{i1}, \cdots, z_{ir})'$ is $r \times 1$ random vector with uncorrelated components, and Var$(z_{i1}) > \cdots >$ Var(Z_{ir}). Furthermore $\mathcal{M}(\mathbf{A}_1) = \mathcal{M}(\mathbf{A}_2)$. Then $z_{1j} = \pm z_{2j}$ for $1 \le j \le r$.

In practice, we use the PCA-ed factor $\widehat{\mathbf{x}}_t$.

The number of non-zero elements of the j-th column of $\widehat{\mathbf{A}}$ is the number of the components of $\mathbf{y}_t$ whose dynamics depends on the j-th factor $\widehat{x}_{tj}$.

Nonstationary factors

C1. $\boldsymbol{\varepsilon}_t \sim \mathrm{WN}(\boldsymbol{\mu}_{\boldsymbol{\varepsilon}}, \boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}})$, $\mathbf{c}'\mathbf{x}_t$ is not white noise for any constant $\mathbf{c} \in \mathcal{R}^p$. Furthermore $\mathbf{A}'\mathbf{A} = \mathbf{I}_r$.

Let $\mathbf{B} = (\mathbf{b}_1, \cdots, \mathbf{b}_{p-r})$ be a $p \times (p-r)$ matrix such that

$(\mathbf{A}, \mathbf{B})$ is a $p \times p$ orthogonal matrix, i.e.

$$\mathbf{B}^\tau \mathbf{A} = 0, \qquad \mathbf{B}^\tau \mathbf{B} = \mathbf{I}_{p-r}.$$

Since $\mathbf{y}_t = \mathbf{A}\mathbf{x}_t + \boldsymbol{\varepsilon}_t$,

$$\mathbf{B}^\tau \mathbf{y}_t = \mathbf{B}^\tau \boldsymbol{\varepsilon}_t$$

i.e. $\{\mathbf{B}^\tau \mathbf{y}_t,\ t = 0, \pm 1, \cdots\}$ is WN.

Therefore

$$\mathrm{Corr}(\mathbf{b}_i^\tau \mathbf{y}_t, \mathbf{b}_j^\tau \mathbf{y}_{t-k}) = 0 \quad \forall\, 1 \le i, j \le p-r \text{ and } k \ge 1.$$

Search for mutually orthogonal directions $\mathbf{b}_1, \mathbf{b}_2, \cdots$ one by one such that the projection of $\mathbf{y}_t$ on each of those directions is a white noise.

Stop the search when such a direction is no longer available, and take $p - k$ as the estimated value of r, where k is the number of directions obtained in the search.

Search for mutually orthogonal directions $\mathbf{b}_1, \mathbf{b}_2, \cdots$ one by one such that the projection of $\mathbf{y}_t$ on each of those directions is a white noise.

Stop the search when such a direction is no longer available, and take $p - k$ as the estimated value of r, where k is the number of directions obtained in the search.

See Pan and Yao (2008) for further details, and also some (preliminary) asymptotic results.

Stationary models

C2. $\mathbf{x}_t$ is stationary, and $\mathrm{Cov}(\mathbf{x}_t, \boldsymbol{\varepsilon}_{t+k}) = 0$ for any $k \geq 0$.

Put $\boldsymbol{\Sigma}_y(k) = \mathrm{Cov}(\mathbf{y}_{t+k}, \mathbf{y}_t)$, $\boldsymbol{\Sigma}_x(k) = \mathrm{Cov}(\mathbf{x}_{t+k}, \mathbf{x}_t)$, $\boldsymbol{\Sigma}_{x\varepsilon}(k) = \mathrm{Cov}(\mathbf{x}_{t+k}, \boldsymbol{\varepsilon}_t)$. By $\mathbf{y}_t = \mathbf{A}\mathbf{x}_t + \boldsymbol{\varepsilon}_t$,

$$\boldsymbol{\Sigma}_y(k) = \mathbf{A}\boldsymbol{\Sigma}_x(k)\mathbf{A}' + \mathbf{A}\boldsymbol{\Sigma}_{x\varepsilon}(k), \qquad k \geq 1.$$

For a prescribed integer $k_0 \geq 1$, define

$$\mathbf{M} = \sum_{k=1}^{k_0} \boldsymbol{\Sigma}_y(k)\boldsymbol{\Sigma}_y(k)'.$$

Then $\mathbf{M}\mathbf{B} = 0$, i.e. the columns of $\mathbf{B}$ are the eigenvectors of $\mathbf{M}$ corresponding to zero-eigenvalues.

Hence under C1 and C2,

Hence under C1 and C2,

The factor loading space $\mathcal{M}(\mathbf{A})$ are spanned by the eigenvectors of $\mathbf{M}$ corresponding to its non-zero eigenvalues, and the number of the non-zero eigenvalues is r.

Hence under C1 and C2,

> The factor loading space $\mathcal{M}(\mathbf{A})$ are spanned by the eigenvectors of $\mathbf{M}$ corresponding to its non-zero eigenvalues, and the number of the non-zero eigenvalues is r.

Let $\widehat{\mathbf{M}} = \sum_{k=1}^{k_0} \widehat{\mathbf{\Sigma}}_y(k)\widehat{\mathbf{\Sigma}}_y(k)'$, where $\widehat{\mathbf{\Sigma}}_y(k)$ denotes the sample covariance matrix of $\mathbf{y}_t$ at lag k.

Hence under C1 and C2,

The factor loading space $\mathcal{M}(\mathbf{A})$ are spanned by the eigenvectors of $\mathbf{M}$ corresponding to its non-zero eigenvalues, and the number of the non-zero eigenvalues is r.

Let $\widehat{\mathbf{M}} = \sum_{k=1}^{k_0} \widehat{\mathbf{\Sigma}}_y(k)\widehat{\mathbf{\Sigma}}_y(k)'$, where $\widehat{\mathbf{\Sigma}}_y(k)$ denotes the sample covariance matrix of $\mathbf{y}_t$ at lag k.

$\widehat{r}$: No. of non-zero eigenvalues of $\widehat{\mathbf{M}}$,

$\widehat{\mathbf{A}}$: its columns are the $\widehat{r}$ orthonormal eigenvectors of $\widehat{\mathbf{M}}$ corresponding to its $\widehat{r}$ largest eigenvalues.

Bootstrap test for r

Note that $r = r_0$ iff the $(r_0 + 1)$-th largest eigenvalue of M is 0 and the r_0-th largest eigenvalue is nonzero.

Consider the testing for $H_0 : \lambda_{r_0+1} = 0$,

We reject H_0 if $\widehat{\lambda}_{r_0+1} > l_\alpha$.

Bootstrap to determine l_α:

1. Compute $\widehat{\mathbf{y}}_t$ with $\widehat{r} = r_0$. Let $\widehat{\boldsymbol{\varepsilon}}_t = \mathbf{y}_t - \widehat{\mathbf{y}}_t$.
2. Let $\mathbf{y}_t^* = \widehat{\mathbf{y}}_t + \boldsymbol{\varepsilon}_t^*$, where $\boldsymbol{\varepsilon}_t^*$ are drawn independently (with replacement) from $\{$

Asymptotics I: $n \to \infty$ and p fixed

(i) $\mathbf{y}_t$ is strictly stationary, $E||\mathbf{y}_t||^{4+\delta} < \infty$ for some $\delta > 0$.

(ii) $\mathbf{y}_t$ is α-mixing satisfying $\sum_j \alpha(j)^{\frac{\delta}{2+\delta}} < \infty$.

(iii) $\mathbf{M}$ has r non-zero eigenvalues $\lambda_1 > \cdots > \lambda_r > 0$.

Then under condition C1 and C2, the following assertions hold.

(i) $\widehat{\lambda}_j - \lambda_j = O_P(n^{-1/2})$ for $1 \le j \le r$,

(ii) $\widehat{\lambda}_{r+k} = O_P(n^{-1})$ for $1 \le k \le p - r$,

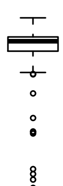
(iii) $D\{\mathcal{M}(\widehat{\mathbf{A}}),\ \mathcal{M}(\mathbf{A})\} = O_P(n^{-1/2})$ provided $\widehat{r} = r$ a.s.,

where

$$D\{\mathcal{M}(\widehat{\mathbf{A}}),\ \mathcal{M}(\mathbf{A})\} = 1 - \frac{1}{r}\text{tr}(\mathbf{A}\mathbf{A}^\tau \widehat{\mathbf{A}}\widehat{\mathbf{A}}^\tau).$$

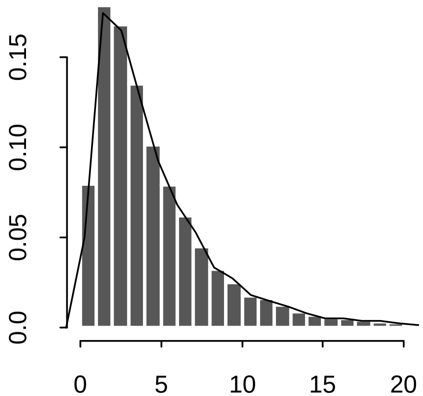
Numerical illustration: $\lambda_1 = 1.884,\ \lambda_2 = \lambda_3 = \lambda_4 = 0$ ($p = 4, r = 1$)

(Simulation replications: 10,000 times)

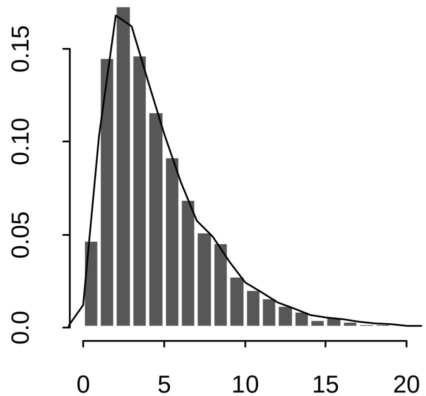


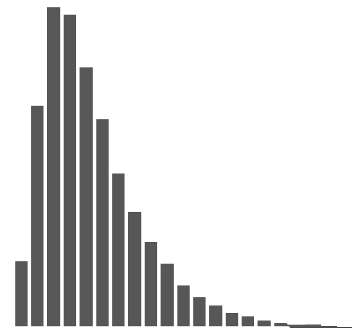
Histogram of $n\widehat{\lambda_2}$

n=10



n=20





Asymptotics II: $n \to \infty$, $p \to \infty$ and r fixed

Recall model: $\mathbf{y}_t = \mathbf{A}\mathbf{x}_t + \boldsymbol{\varepsilon}_t$, and $\mathbf{A}$ is $p \times r$

1. Assumptions on Strength of factors:

   (i) $\mathbf{A} = (\mathbf{a}_1, \cdots, \mathbf{a}_r)$, $||\mathbf{a}_i||^2 \asymp p^{1-\delta}, \quad i = 1, \cdots, r, \quad 0 \le \delta \le 1.$

   (ii) For $k = 0, 1, \cdots, k_0$, $\boldsymbol{\Sigma}_x(k) \equiv \mathrm{Cov}(\mathbf{x}_{t+k}, \mathbf{x}_t)$ is full-ranked, and $\boldsymbol{\Sigma}_{x,\epsilon}(k) \equiv \mathrm{Cov}(\mathbf{x}_{t+k}, \boldsymbol{\varepsilon}_t) = O(1)$ elementwisely.

We call

- factors are strong if $\delta = 0$,
- factors are weak if $\delta > 0$.

Standardization '$\mathbf{A}^\tau \mathbf{A} = \mathbf{I}_r$' + (i, ii) imply:

$$\|\boldsymbol{\Sigma}_x(k)\| \asymp p^{1-\delta} \asymp \|\boldsymbol{\Sigma}_x(k)\|_{\min}, \quad \|\boldsymbol{\Sigma}_{x,\epsilon}(k)\| = O(p^{1-\delta/2}),$$

where $a \asymp b$ represents $a = O(b)$ & $b = O(a)$, $\|\mathbf{A}\|^2 = \lambda_{\max}(\mathbf{A}\mathbf{A}^\tau)$ and $\|\mathbf{A}\|^2_{\min} = \min\{\lambda(\mathbf{A}\mathbf{A}^\tau) : \lambda(\mathbf{A}\mathbf{A}^\tau) > 0\}$.

2. For $k = 0, 1, \cdots, k_0$, $\|\mathbf{\Sigma}_{x,\epsilon}(k)\| = o(p^{1-\delta})$, and it holds elementwisely that

$$\tilde{\mathbf{\Sigma}}_x(k) - \mathbf{\Sigma}_x(k) = O_P(n^{-l_x}), \quad \tilde{\mathbf{\Sigma}}_\epsilon(k) - \mathbf{\Sigma}_\epsilon(k) = O_P(n^{-l_\epsilon}),$$

$$\tilde{\mathbf{\Sigma}}_{x,\epsilon}(k) - \mathbf{\Sigma}_{x,\epsilon}(k) = O_P(n^{-l_{x\epsilon}}) = \tilde{\mathbf{\Sigma}}_{\epsilon,x}(k)$$

   for some constants $0 < l_x, l_{x\epsilon}, l_\epsilon \leq 1/2$, and $\tilde{\mathbf{\Sigma}}$ denotes the sample version of $\mathbf{\Sigma}$.

3. $\mathbf{M}$ has r different non-zero eigenvalues.

Then under condition C1 and C2,

$$\|\widehat{\mathbf{A}} - \mathbf{A}\| = O_P(h_n) = O_P(n^{-l_x} + p^{\delta/2} n^{-l_{x\epsilon}} + p^\delta n^{-l_\epsilon}),$$

provided $h_n = o(1)$.

Remark. When all factors are strong (i.e. $\delta = 0$), the convergence rate h_n is independent of the dimension p.

Our asymptotic theory also shows:

1. Factor model-based estimator for $\mathbf{\Sigma}_y$:

$$\widehat{\mathbf{\Sigma}}_y = \widehat{\mathbf{A}}\widehat{\mathbf{\Sigma}}_x\widehat{\mathbf{A}}^\tau + \widehat{\mathbf{\Sigma}}_\epsilon, \quad \text{where} \quad \widehat{\mathbf{\Sigma}}_x = \widehat{\mathbf{A}}^\tau(\tilde{\mathbf{\Sigma}}_y - \widehat{\mathbf{\Sigma}}_\epsilon)\widehat{\mathbf{A}},$$

cannot improve over the sample covariance estimator $\tilde{\mathbf{\Sigma}}_y$.

But the convergence rate for $\|\widehat{\mathbf{\Sigma}}_y^{-1} - \mathbf{\Sigma}_y^{-1}\|$ is independent of p when all the factors are strong.

Simulation with $r = 1$ and $\delta = 0$ (only one strong factor):

$$x_t = 0.9 x_{t-1} + N(0, 4),$$

$\varepsilon_{tj} \sim_{iid} N(0, 4)$, and the i-th element of $\mathbf{A}$ is $2\cos(2\pi i/p)$.

$n = 200$	$\lVert\widehat{\mathbf{A}} - \mathbf{A}\rVert$	$\lVert\tilde{\mathbf{\Sigma}}_y^{-1} - \mathbf{\Sigma}_y^{-1}\rVert$	$\lVert\widehat{\mathbf{\Sigma}}_y^{-1} - \mathbf{\Sigma}_y^{-1}\rVert$
$p = 20$	$.022(.005)$	$.24(.03)$	$.009(.002)$
$p = 180$	$.023(.004)$	$79.8(29.8)$	$.007(.001)$
$p = 400$	$.022(.004)$	-	$.007(.001)$
$p = 1000$	$.023(.004)$	-	$.007(.001)$

n

Illustration With Real Data

Example 1. The monthly temperature data from 7 cities in Eastern China in January 1954 — December 1986

$$n = 396, \quad p = 7, \quad \widehat{r} = 4$$

Example 2. Daily implied volatility surfaces for IBM, Microsoft and Dell call options in 2006

$$n = 100, \quad p = 130, \quad \widehat{r} = 1$$

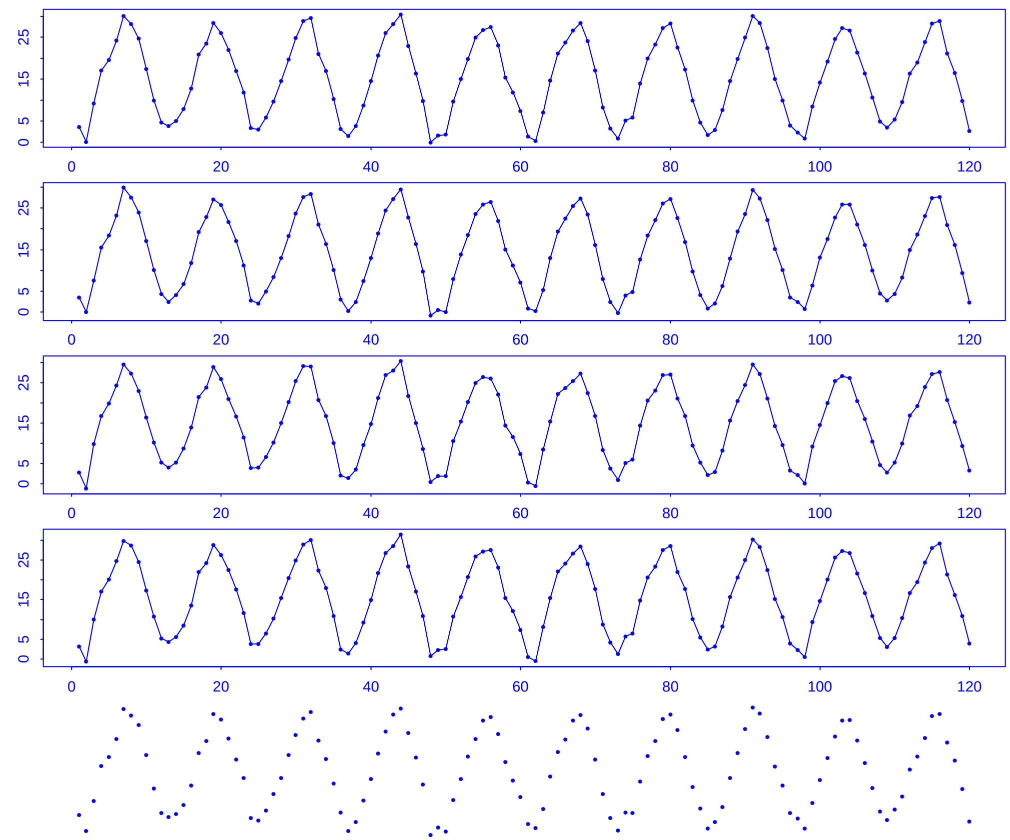
⇛

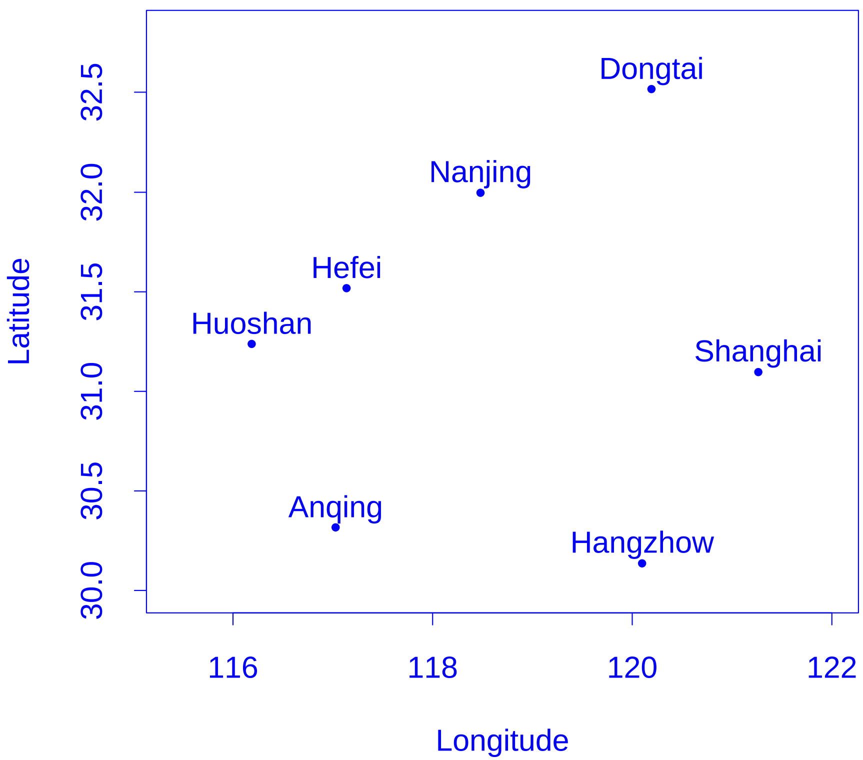
Example 3. Daily densities of one-minute returns of IBM stock price in 2006

$$n = 251, \quad p = \text{“}\infty\text{”}, \quad \widehat{r} = 2$$

⇛

Time plots of the monthly temperature in 1959-1968 of Nanjing, Dongtai, Huoshan, Hefei, Shanghai, Anqing and Hangzhow.





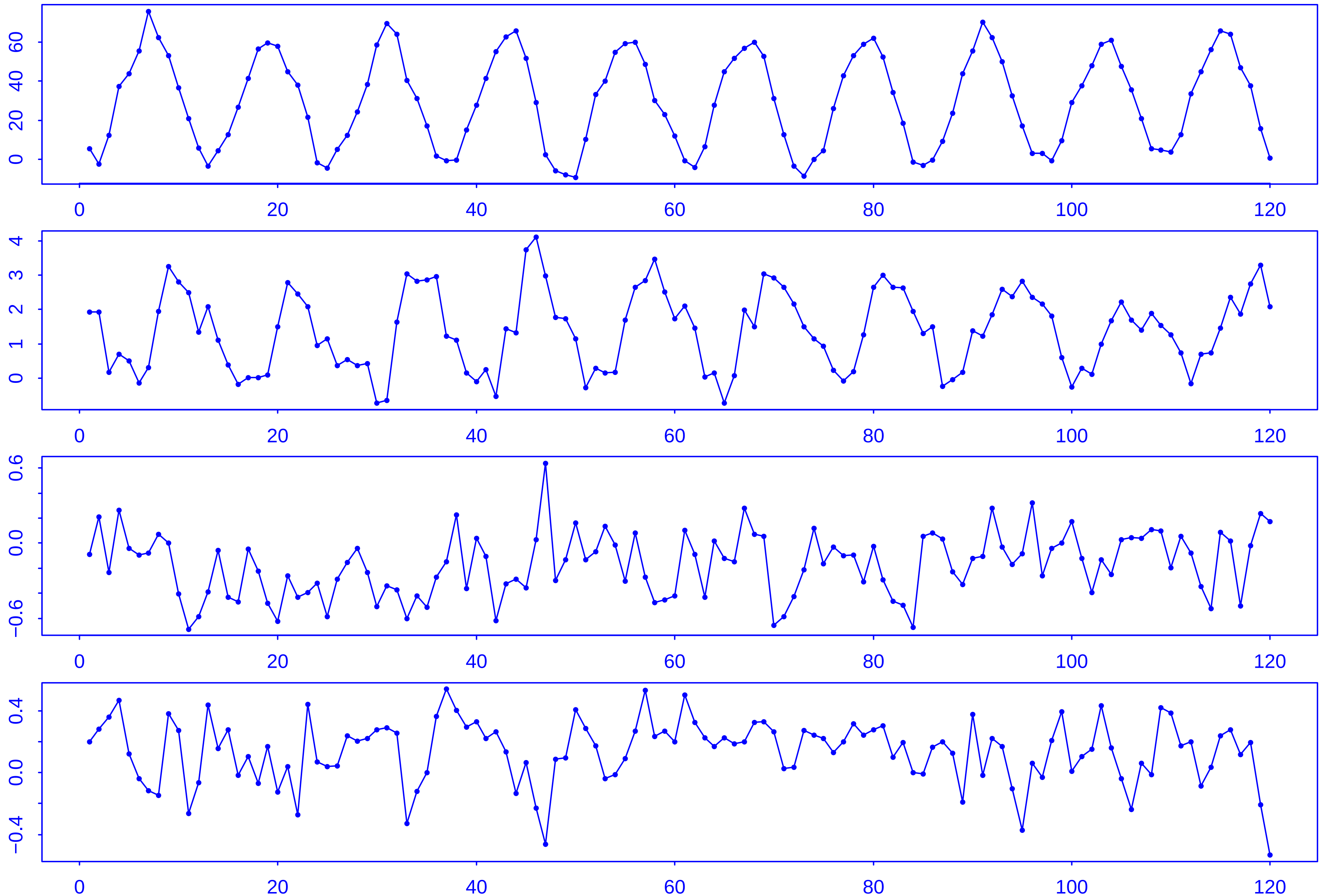
With $p = 12$, $\alpha = 1\%$, the fitted model is $\mathbf{y}_t = \widehat{\mathbf{A}}\mathbf{x}_t + \mathbf{e}_t$, $\widehat{r} = 4$,
$\mathbf{e}_t \sim \mathrm{WN}(\widehat{\boldsymbol{\mu}}_\varepsilon, \widehat{\boldsymbol{\Sigma}}_\varepsilon)$,

$$\widehat{\boldsymbol{\mu}}_e = \begin{pmatrix} 3.41 \\ 2.32 \\ 4.39 \\ 4.30 \\ 3.40 \\ 4.91 \\ 4.77 \end{pmatrix}, \quad \widehat{\boldsymbol{\Sigma}}_e = \begin{pmatrix} 1.56 & & & & & & \\ 1.26 & 1.05 & & & & & \\ 1.71 & 1.34 & 1.91 & & & & \\ 1.90 & 1.49 & 2.10 & 2.33 & & & \\ 1.37 & 1.16 & 1.46 & 1.58 & 1.37 & & \\ 1.67 & 1.26 & 1.91 & 2.09 & 1.37 & 1.97 & \\ 1.41 & 1.14 & 1.58 & 1.67 & 1.39 & 1.56 & 1.53 \end{pmatrix}.$$

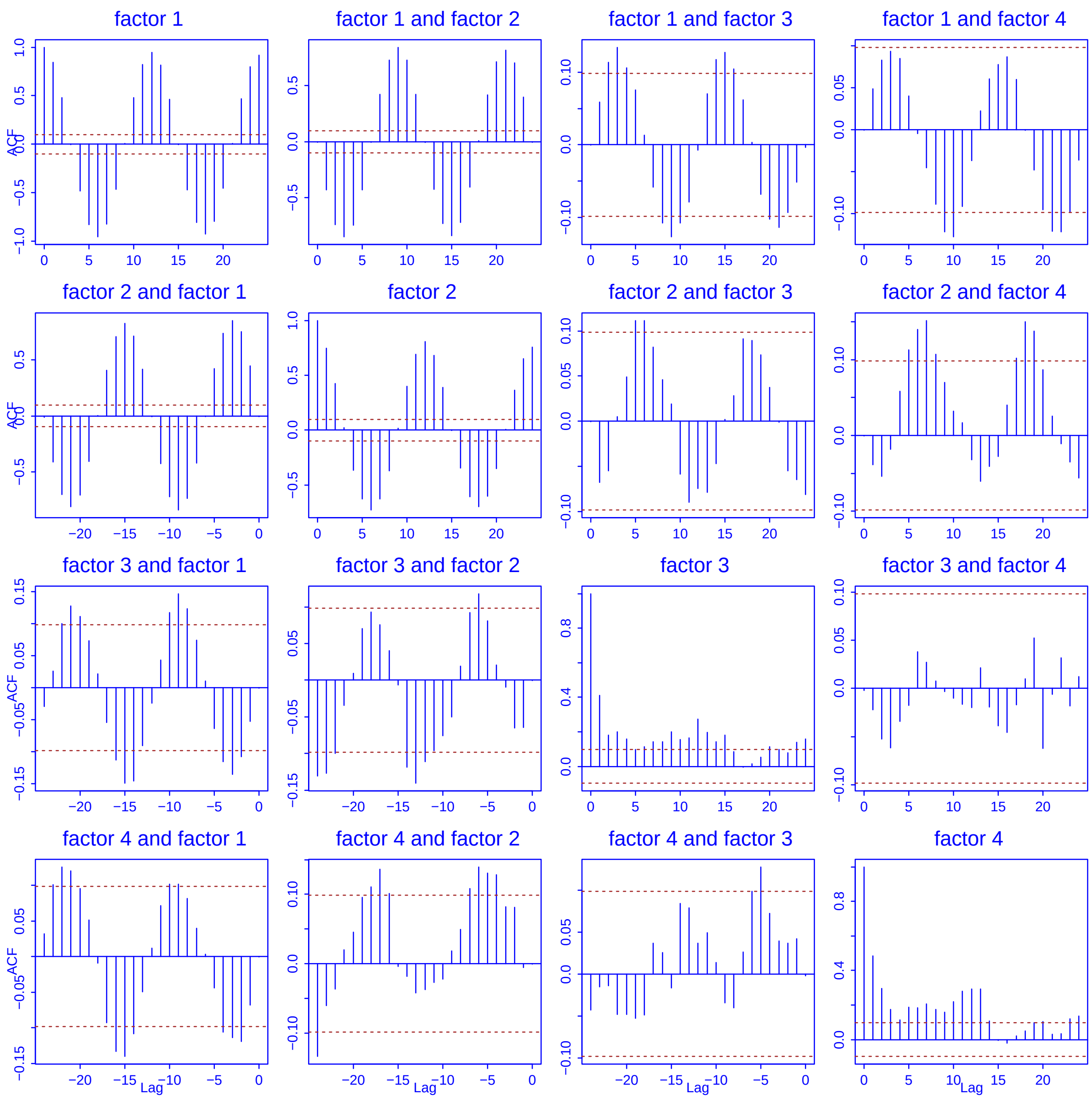
$$\widehat{\mathbf{A}} = \begin{pmatrix} .394 & .386 & .378 & .387 & .363 & .376 & .366 \\ -.086 & .225 & -.640 & -.271 & .658 & -.014 & .164 \\ .395 & .0638 & -.600 & .346 & -.494 & -.074 & .332 \\ .687 & -.585 & -.032 & -.306 & .173 & .206 & -.139 \end{pmatrix}^{\tau},$$

$\mathbf{x}_t$ are PCAed factors: 1st PC accounts for 99% of TV of 4 factors, and 97.6% of the original 7 series.

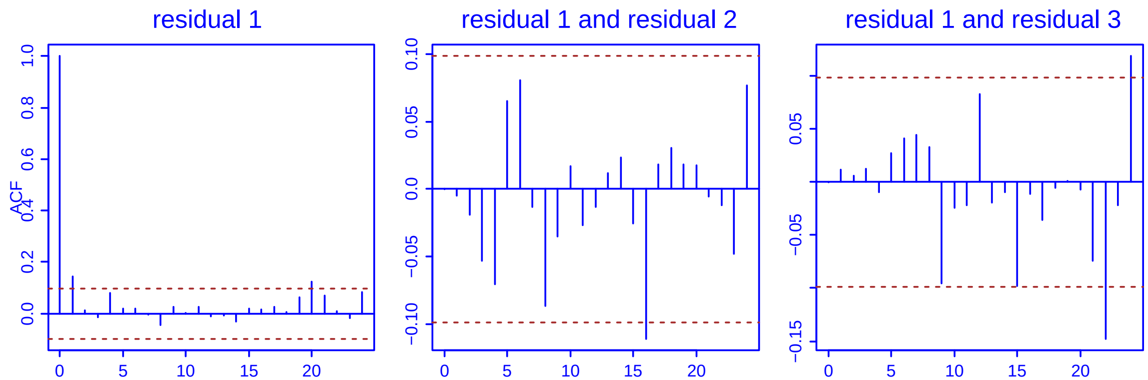
Time plots of the 4 estimated factors VAR(1)



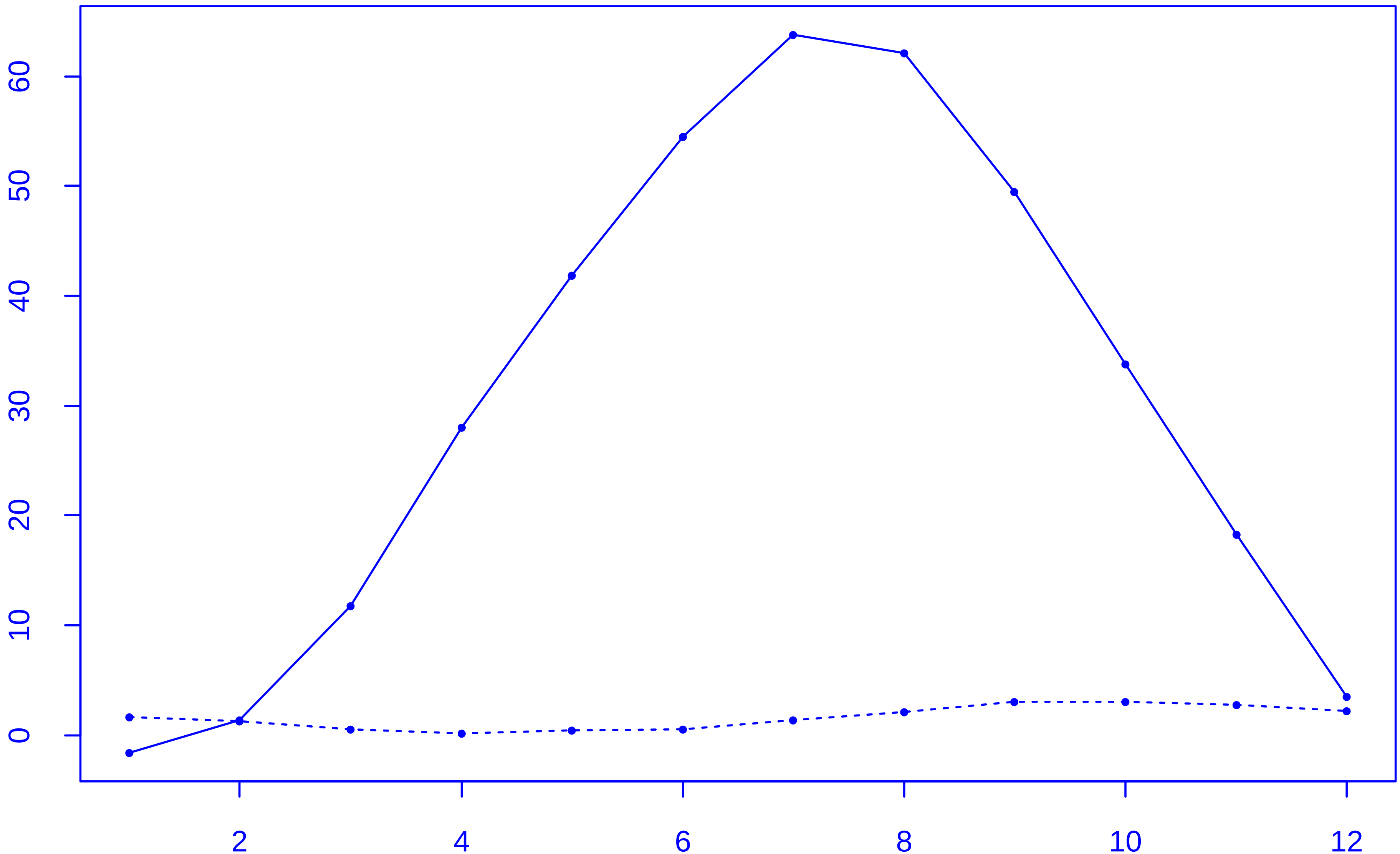
Sample cross-correlation of the 4 estimated factors



Sample cross-correlation of the 3 residuals (i.e. $\widehat{\mathbf{B}}^\tau \mathbf{y}_t$)



Since the first two factors are dominated by periodic components, we remove them before fitting.



In the fitted factor model $\mathbf{y}_t = \widehat{\mathbf{A}}\mathbf{x}_t + \mathbf{e}_t$, the AICC selected VAR(1) for the

- Temperature dynamics in the 7 cities may be modelled in terms of 4 common factors
- The annual periodic fluctuations may be explained by a single common factor
- Removing the periodic components, the dynamics of the 4 common factors may be represented by an AR(1) model

⇛

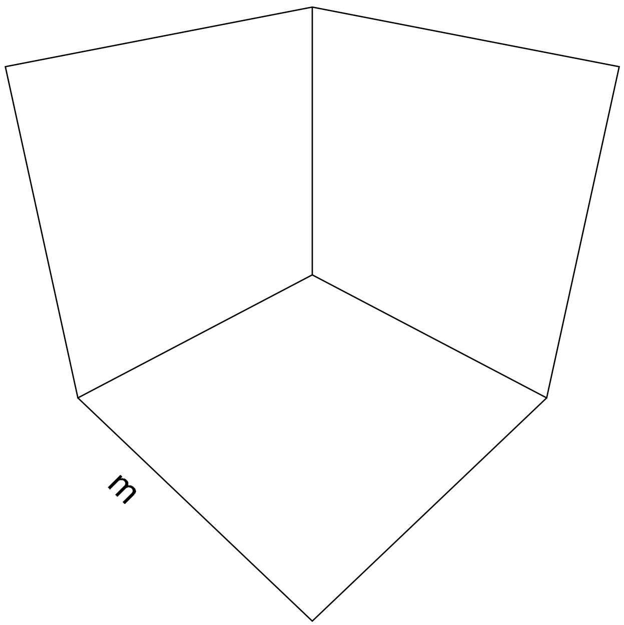
Example 2. Implied volatility surfaces of IBM, Microsoft and Dell stocks in 2006 (i.e. 251 trading days).

Source of Data: OptionMetrics at WRDS

Observations: for $t = 1, \cdots, 251$, implied volatility $w_t(u_i, v_j)$ computed from call options at

- time to maturity at 30, 60, 91, 122, 152, 182, 273, 365, 547 & 730 calendar days, denoted by $u_1, \cdots, u_{10}$, and
- delta at 0.2, 0.25, 0.3, 0.35, 0.4, $\cdots$, 0.8, denoted by $v_1, \cdots, v_{13}$.

Total: $p = 10 \times 13 = 130$ wt w_t


m

Fitting a factor model on each of the rolling windows of length 100 days:

$$\mathbf{y}_i, \mathbf{y}_{i+1}, \cdots, \mathbf{y}_{i+99}, \quad i = 1, \cdots, 150.$$

The estimated number of factors for all 3 stocks across different windows is always $\widehat{r} = 1$.

Based on a fitted AR model to the estimated factor process, we predict the next value x_{i+100}, denoted by $\check{x}_{i+100}$. It leads to the one-step ahead prediction for $\mathbf{y}_{i+100}$:

$$\check{\mathbf{y}}_{i+100} = \widehat{\mathbf{A}}\check{x}_{i+100}.$$

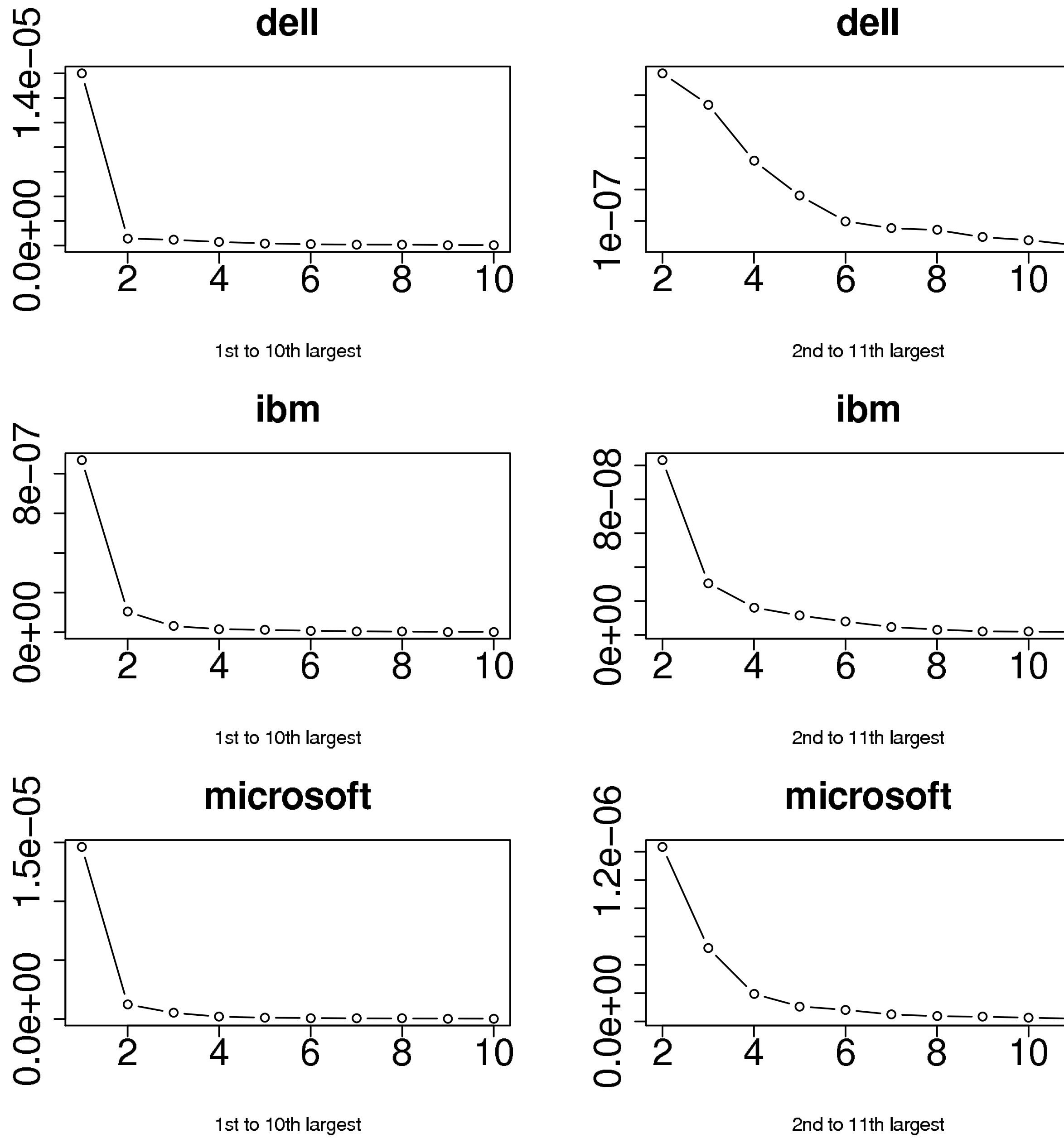
Put

$$\mathrm{RMSE}_i = \frac{1}{\sqrt{p}}||\check{\mathbf{y}}_{i+100} - \mathbf{y}_{i+100}||, \quad i = 1, \cdots, 150.$$

Average of the ordered eigenvalues of $\widehat{\mathrm{M}}$ over the 150 rolling windows.

3 panels on the left: 10 largest eigenvalues

3 panels on the right: 2nd–11th largest eigenvalues.

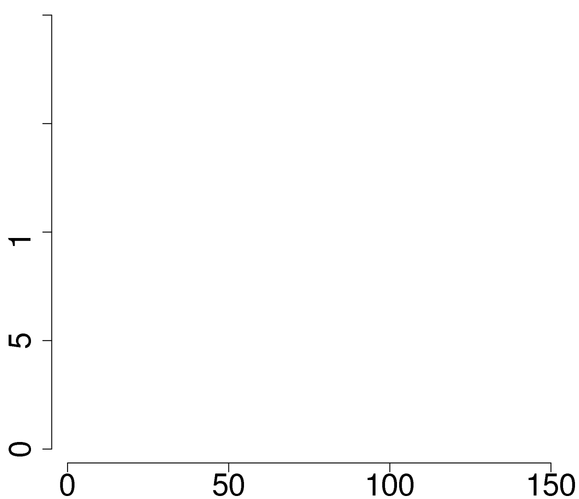


Benchmark prediction for $\mathbf{y}_{i+100}$: the previous value $\mathbf{y}_{i+99}$

Prediction based on Bai & Ng (2002) — factor-modelling based on the LSE: $(\widehat{\mathbf{A}}, \widehat{\mathbf{x}}_t)$ is the solution of

$$\min_{\mathbf{A},\mathbf{x}_t} \sum_{t=1}^{n} ||\mathbf{y}_t - \mathbf{A}\mathbf{x}_t||^2, \quad \text{subject to } \mathbf{A}^\tau\mathbf{A}/p = \mathbf{I}_r \text{ and } \mathbf{X}^\tau\mathbf{X}/n = \mathbf{I}_r,$$

where $\mathbf{X} = (\mathbf{x}_1, \cdots, \mathbf{x}_n)$.



Example 3. IBM stock intra-day prices in 2006

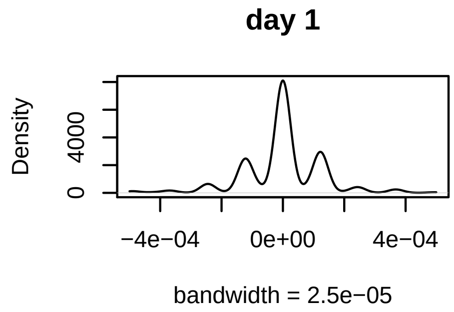
251 trading days, tick by tick prices collected in 9:30 — 16:00

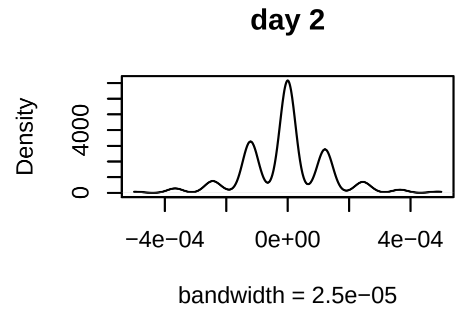
In total 2,786,649 observations (74MB)

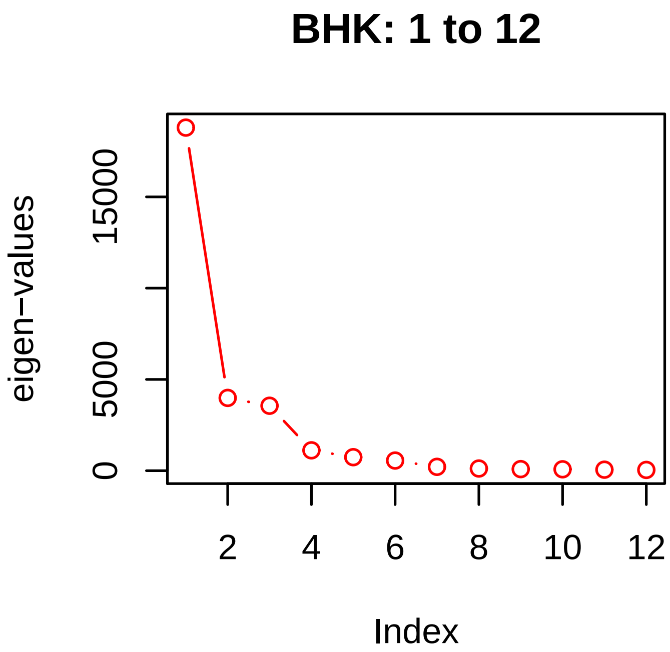
For each of 251 trading days, construct the pdf curve of one-minute log-return using the log-returns in 390 one-minute intervals: kernel density estimation with $h = 0.000025$

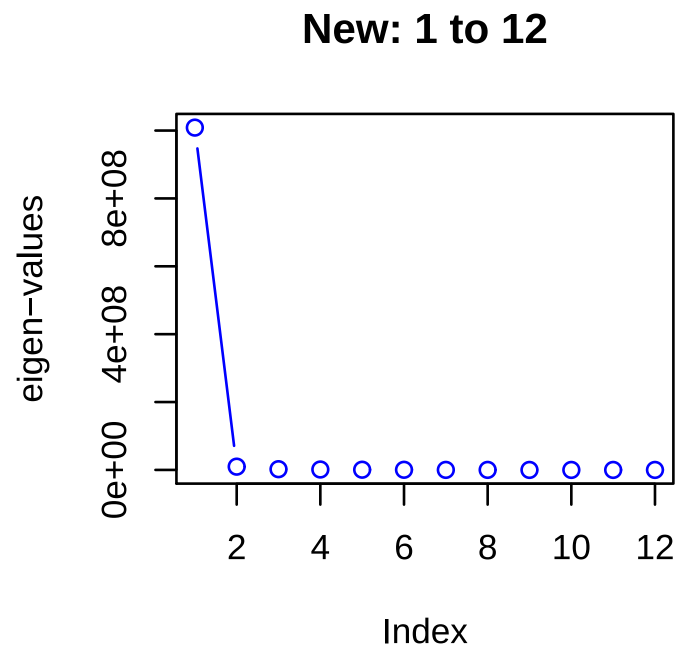
Treating the 251 pdfs as a high-dimensional time series, apply the proposed procedure.

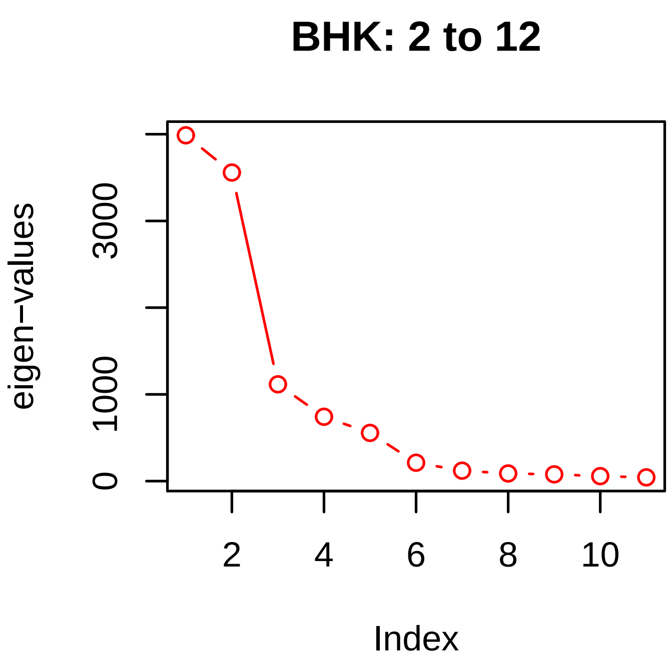
The white-noise test rejects $H_0 : r = 1$, but cannot reject $H_0 : r = 2$.

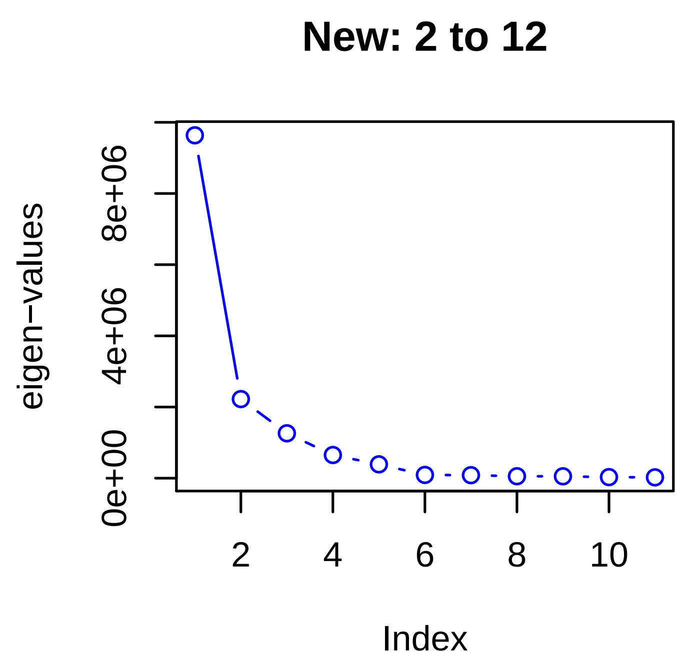

day 1
Density
0
4000
−4e−04
0e+00
4e−04
bandwidth = 2.5e−05

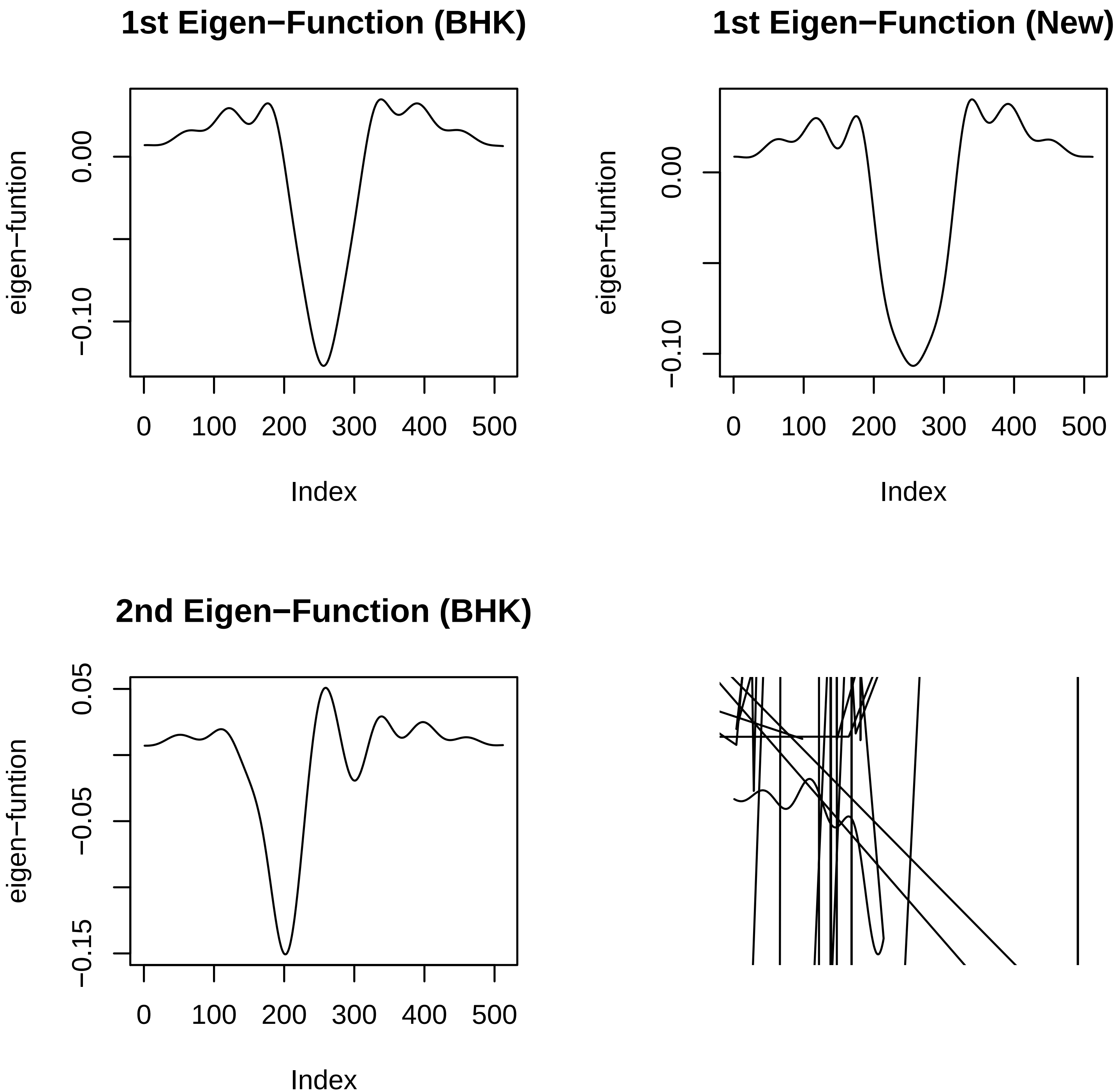

day 2
Density
0
4000
−4e−04
0e+00
4e−04
bandwidth = 2.5e−05


BHK: 1 to 12
eigen−values
0
5000
15000
2
4
6
8
10
12
Index

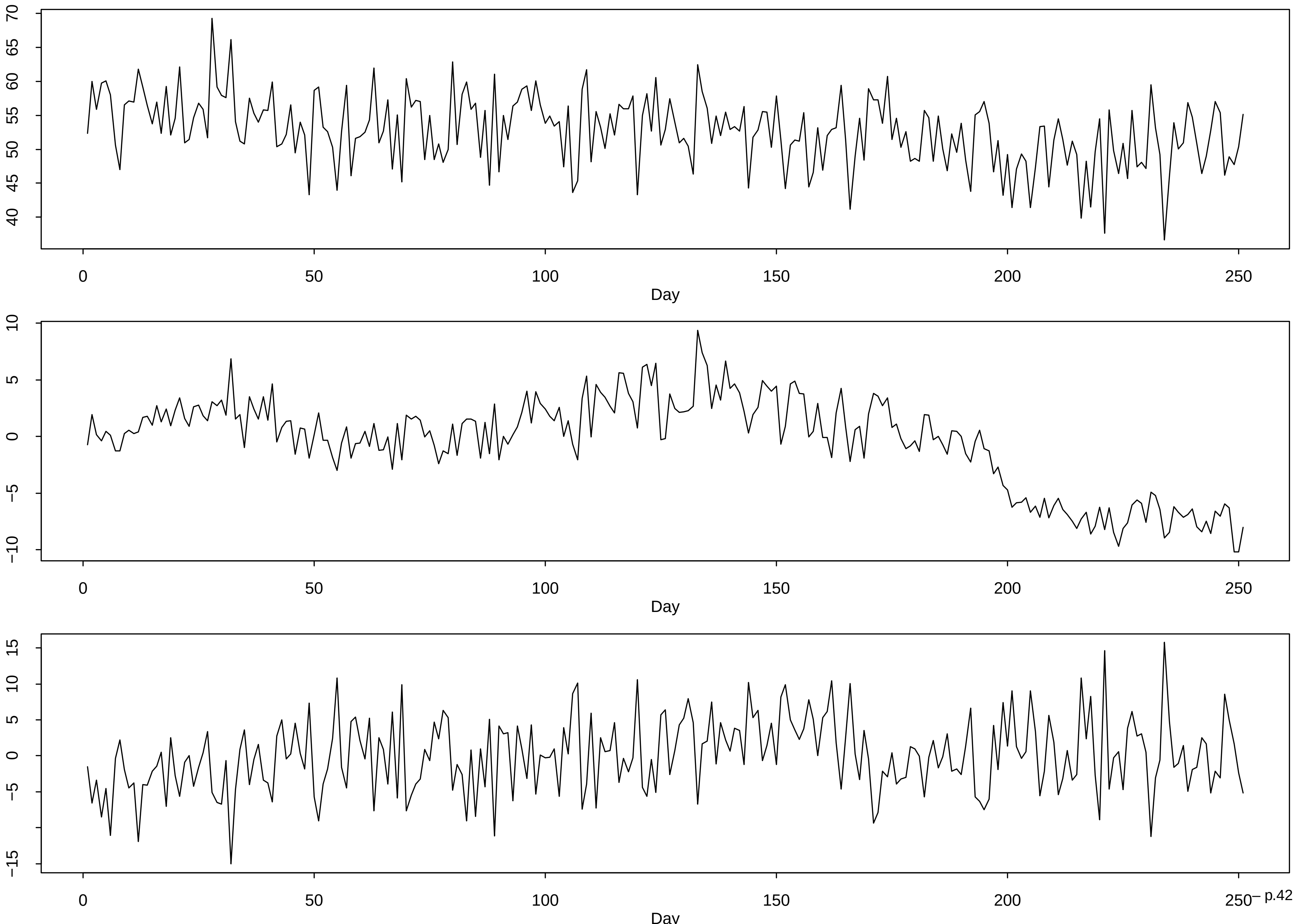

New: 1 to 12
eigen−values
0e+00
4e+08
8e+08
2
4
6
8
10
12
Index


BHK: 2 to 12
eigen−values
0
1000
3000
2
4
6
8
10
Index

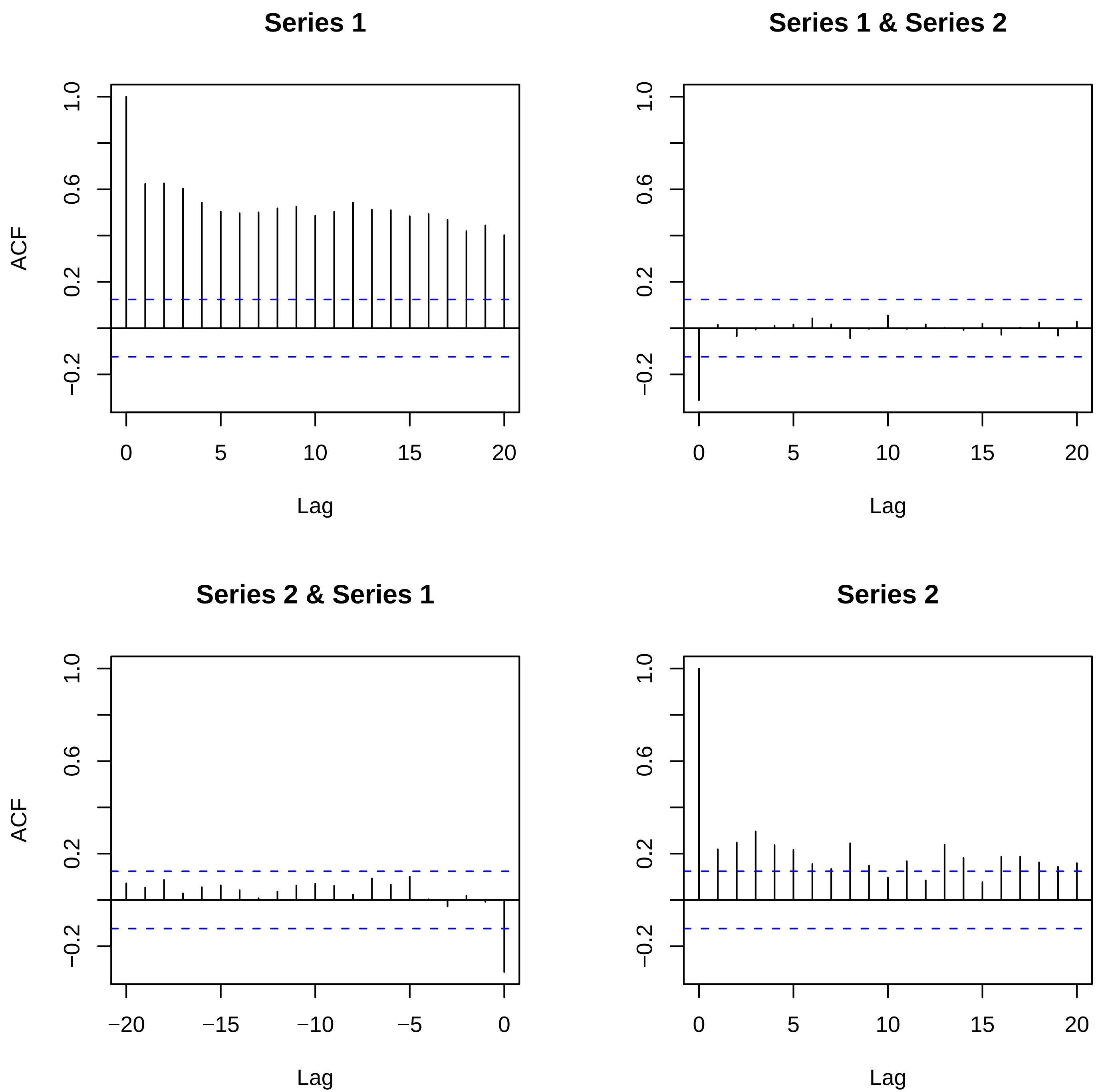

New: 2 to 12
eigen−values
0e+00
4e+06
8e+06
2
4
6
8
10
Index


1st Eigen−Function (BHK)
eigen−funtion
0.00
−0.10
0
100
200
300
400
500
Index
1st Eigen−Function (New)
eigen−funtion
0.00
−0.10
0
100
200
300
400
500
Index
2nd Eigen−Function (BHK)
eigen−funtion
0.05
−0.05
−0.15
0
100
200
300
400
500
Index

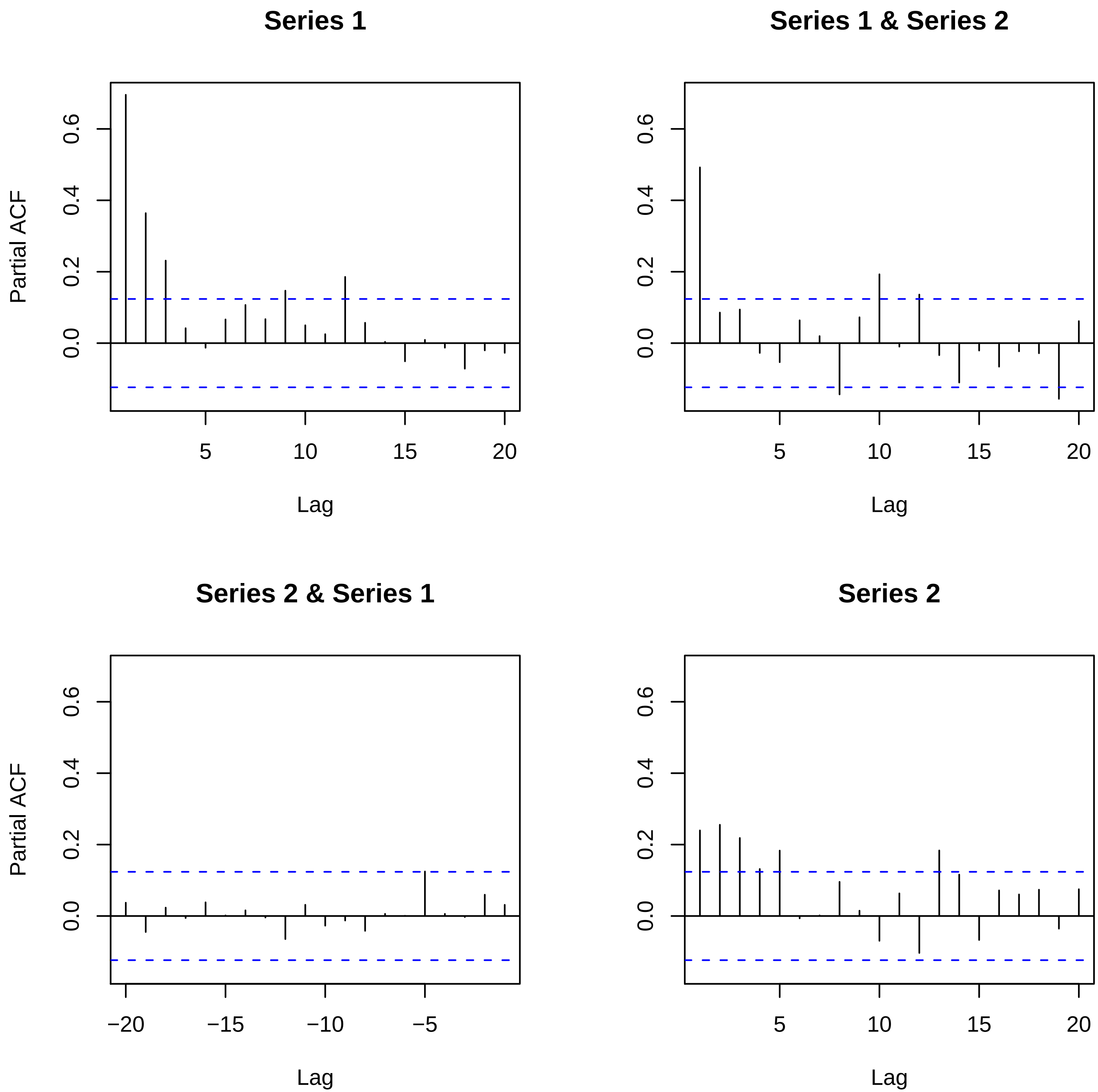
Time series plots of x_{t1} and x_{t2}



ACF of (x_{t1}, x_{t2})



PACF of (x_{t1}, x_{t2})



Fitting time series $\mathbf{x}_t = (x_{t1},\ x_{t2})'$

Since there is little cross correlation between the two component series, we fit them separately.

For $\{x_{t1}\}$, AIC selected ARMA(1,1) with AIC=4556.76:

$$x_{t+1,1} = 0.985 x_{t1} + \varepsilon_{t+1,1} - 0.787 \varepsilon_{t,1}.$$

For $\{x_{t2}\}$, AIC selected ARMA(1,1) with AIC=4323.1:

$$x_{t+1,2} = 0.982 x_{t2} + \varepsilon_{t+1,2} - 0.885 \varepsilon_{t,2}.$$

Allowing nonstationarity — ARIMA(1,1,1):

$$x_{t+1,1} - x_{t1} = 0.062(x_{t1} - x_{t-1,1}) + \varepsilon_{t+1,1} - 0.847 \varepsilon_{t,1}, \quad (\text{AIC} = 4537.13)$$

$$x_{t+1,2} - x_{t2} = 0.046(x_{t2} - x_{t-1,2}) + \varepsilon_{t+1,2} - 0.889 \varepsilon_{t,2}, \quad (\text{AIC} = 4306.08)$$

ACF of the residuals from the fitted ARMA(1,1) models